

# Hydrodynamic Description of Granular Convection

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## Abstract

We present a hydrodynamic model that captures the essence of granular dynamics in a vibrating bed. We carry out the linear stability analysis and uncover the instability mechanism that leads to the appearance of the convective rolls via a supercritical bifurcation of a bouncing solution. We also explicitly determine the onset of convection as a function of control parameters and confirm our picture by numerical simulations of the continuum equations.

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Granular materials in a container subjected to vertical vibrations display interesting nonlinear dynamical behaviors. [1–6] Nothing really happens for  $\Gamma = A\omega^2/g < 1$ , where  $A$  and  $\omega$  are the frequency and the amplitude of the oscillations and  $g$  is the gravitational constant. But for  $1 < \Gamma$ , the granular materials collectively move up and down, which we term the uniform bouncing [7], until  $\Gamma$  reaches the critical value  $\Gamma_c$  beyond which such a uniform bouncing motion becomes unstable and the permanent convective rolls develop inside the bulk. [3–5] Recent studies have revealed further complexity of this problem for values of  $\Gamma$  much larger than one, where reverse convection [5] and bubble formation [6] have been

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observed. Current efforts to understand the experiments of granular dynamics [3,5,6] have mostly focused on large scale Molecular Dynamics (MD) simulations. [4] While successful in reproducing convection cells and some of the experimental results, such studies have limitations in understanding the analytic structure of the instability mechanism and/or its subsequent dynamic evolution. There have been a handful of attempts in the past to derive continuum equations for granular dynamics notably by Jenkins and Savage for rapid granular flow problems [8] and by Haff for vibrating beds [9], but these studies have been mostly confined either to simple cases of one dimensional oscillations in an infinite system where pressure inside the grains behave like a fluid [9], or to cases where explicit assumption has been made regarding the Gaussian nature in velocity distribution of grains [8], which have been shown to break down in a dense granular system [10]. There also has been a recent attempt by Bourzutchky and Miller [11] who have utilized the Navier Stoke equation along the similar line of Haff [9] and have reproduced numerically the convective rolls. However, we find it difficult to imagine that the hydrodynamic pressure term ( $\rho g z$ ) exists to cancel the gravity term inside the granular materials that undergo vertical vibrations.

The purpose of this Letter is two fold: we first propose a dynamic model that is simple enough to make progress in analytic studies, yet captures, in our opinion, the essence of granular dynamics of vibrating beds. Second, we demonstrate that the correct way of studying the convective instability is to carry out the stability analysis around the *bouncing solution* and explicitly determine the onset of convection as a function of external parameters. We will also present numerical results to confirm our predictions.

**Equations of motion:** Our starting point is the recognition that the most fundamental aspect of the vibrating bed, apart from the obvious fixed bed solution with no external driving, is the existence of a uniform bouncing of a collection of particles. Such a bouncing solution can be either a solid block inside the bed or a fluidized state with a slightly expanded volume yet with no internal degrees of freedom. This assumption is consistent with observations in MD [4] where surface fluidization rapidly spreads out into bulk regions when

surface fluidization is suppressed. In such a case, the bouncing solution can be represented by a motion of a ball on a vibrating platform. For small  $\Gamma$ , no exotic motion such as chaotic motion is expected to occur for such a ball [12]. We further assume that the restitution constant of the collection of particles is zero to represent the relaxation of inside collection of particles. In such a case, the relative position of the ball with respect to the bottom plate,  $\Delta(t)$ , is given by:

$$\Delta(t) = \Gamma(\sin t_0 - \sin t) + \Gamma \cos t_0(t - t_0) - \frac{1}{2}(t - t_0)^2 \quad (1)$$

in the unit of  $g = \omega = 1$ , where the ball starts to bounce at  $t_0$  on the bottom plate, whose position at time  $t$  is given by  $\Gamma \sin t$  in the experimental frame. The bouncing solution is then described by the relative speed between the plate and the ball:  $V_{rel} \equiv d\Delta(t)/dt$ . Since  $\Delta(t)$  cannot be negative, the ball launched upward on the plate at  $t_0$  falls back to the plate at  $t_1$  (i.e.,  $\Delta(t_1) = 0$ ) and stays there until  $t = t_0 + 2\pi$  from our assumption of the zero restitution constant. The ball is then relaunched and obeys (1) again. For later use, we determine  $(t_0 = \sin^{-1}(1/\Gamma), t_1)$  for different values of  $\Gamma$ . For example,  $(t_0, t_1) = (1.181, 2.88225)$  for  $\Gamma = 1.1$  and  $(t_0, t_1) = (0.524, 5.18)$  for  $\Gamma = 2.0$ . When we expand  $\Delta(t)$  around  $t_0$  we obtain  $\Delta(t) \simeq (\Gamma/6) \cos(t_0)(t - t_0)^3 > 0$ , where  $\cos(t_0) > 0$  from the launching condition  $d^2V_{rel}/dt^2 > 0$ . Hence, there is no solution of  $\Delta(t) = 0$  around  $t = t_0$  except for  $t = t_0$ . Therefore, the bouncing motion starts from the finite  $t_1 - t_0$ . One can now readily derive the equation of motion for the vertical coordinate  $z$  for the bouncing motion of a granular block:  $\ddot{z} = (-1 + \Gamma \sin t)\theta(-1 + \Gamma \sin t)$  where the  $\theta(x) = 1$  for  $x > 0$  and  $\theta(x) = 0$  for otherwise.

In order to describe the motion of a granular block in the presence of internal degrees of freedom such as rotation and/or translation, we define two coarse-grained dynamical variables: the density  $\rho(\mathbf{r}, t)$  and the velocity  $\mathbf{v}(\mathbf{r}, t)$  of the granular system. In a box fixed frame,  $\rho$  and  $\mathbf{v}$  then should satisfy the continuity and the momentum equation:

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (2)$$

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \hat{z}(\Gamma \sin t - 1 - \lambda) - \frac{1}{\rho} \nabla P + \frac{1}{R} [\nabla^2 \mathbf{v} + \chi \nabla(\nabla \cdot \mathbf{v})] \quad (3)$$

where  $\hat{z}$  is the unit vector in the vertical direction and  $\lambda$  is a Lagrange multiplier.  $\lambda = 0$  for free motion and  $\lambda = \Gamma \sin t - 1$  for stationary state. Note that the first term in (3) is due to the uniform bouncing and the third term is the energy dissipation effectively represented by the Reynolds number  $R$  and the bulk viscosity  $\chi$ . Now, the exact form of the pressure  $P$  in (3) is unknown for granular materials. Unlike fluid, for granular materials in a container supported by the side walls, the pressure inside the bulk seems to saturate [1,13]. In such a case, the only contribution to the granular pressure would result from the hard sphere repulsion which might be effectively represented by the Van der Waals equation:

$$P = \frac{T\rho}{1 - b\rho} \quad (4)$$

where  $T \approx \langle \mathbf{v}^2 \rangle$  is the granular temperature [9] and  $b$  is a constant of order unity. Note that eqs.(2) and (3) are precisely the *compressible* Navier-Stokes equation with two modifications: first the hydrodynamic pressure term is absent, which is replaced by the Van der Waals form (4), and second, the gravity term thus survives in the vibrating bed and has been effectively modified to  $g - A\omega^2 \sin t$  in the physical unit. Notice that the term  $\Gamma \sin t$  appears since we have used the box-fixed frame. We now analyze eqs.(2) and (3).

**Linear stability analysis:** (a) Fixed bed solution: Fixed bed is a container filled with grains with no external driving. In this case, the contact force balances out the gravity and the net force acting on each grain is zero. So, we use  $\lambda = \Gamma \sin t - 1$  in (3). In this case, the solution with constant density  $\rho = \rho_0$  and zero speed  $\mathbf{v} = 0$  is stable.

(b) Linear stability of a uniform bouncing solution: In order to discuss the stability of the uniform bouncing solution,  $\rho = \rho_0$  and  $\mathbf{v} = (0, 0, V_{rel}(t))$ , against fluctuations, we set  $\rho = \rho_0 + \rho_L$ , and decompose the velocity into the vertical and horizontal components,  $\mathbf{v}_L = (\mathbf{v}_{\perp,L}, w_L)$  with  $\mathbf{v}_{\perp} = \mathbf{v}_{\perp,L}$  and  $w = V_{rel}(t) + w_L$ . We then substitute these into dynamic equations (2) and (3) and introduce a new coordinate to simplify the problem,  $\xi = z - \int^t V_{rel}(t') dt'$ . Upon linearization, we obtain the following equation for the perturbed density  $\rho_L$ :

$$[\partial_t^2 - \frac{1}{\hat{R}} \nabla_\xi^2 \partial_t - T_e \nabla_\xi^2] \rho_L = 0 \quad (5)$$

where  $\hat{R} = R/(1 + \chi)$ . We now solve (5) in 2 dimension under the no current boundary condition at the plate and at  $z = \infty$ , namely:  $\rho_L = 0$  at  $x = 0, L$ , and  $z = 0, \infty$  where  $L$  is the dimensionless size of the box. To satisfy these boundary conditions, we set:

$$\rho_L(x, y, z, t) = \rho_{L,q,m}(t) \sin[\hat{\pi} m x] \sin[q(\xi - S(t))] \quad (6)$$

where  $\hat{\pi} = \pi/L$ ,  $m$  is an integer, and  $S(t) = -\Delta(t)$ . We notice that the spectrum is discrete for  $x$  direction but continuous along  $z$  direction. We now substitute (6) into (5) and utilize the fact,  $t = \tau + t_0$  with  $t_0 = \sin^{-1} 1/\Gamma$ . After some algebra, we obtain the following second order ordinary differential equation for the amplitude  $\rho_q(t) = \rho_{L,q,m}(t)$ :

$$\ddot{\rho}_q + B(q)\dot{\rho}_q + iC(q)\dot{\rho}_q + D(q)\rho_q + iE(q)\rho_q = iL_q(\tau)\dot{\rho}_q + M_q(\tau)\rho_q + iN_q(\tau)\rho_q \quad (7)$$

where

$$B(q) = \hat{R}^{-1}(\hat{\pi}^2 m^2 + q^2), \quad C(q) = 2q\sqrt{\Gamma^2 - 1} \quad (8)$$

$$D(q) = T_e(q^2 + \hat{\pi}^2 m^2) - \frac{3}{2}q^2\Gamma^2 + q^2, \quad E(q) = -q + \sqrt{\Gamma^2 - 1}\hat{R}^{-1}q(\hat{\pi}^2 m^2 + q^2) \quad (9)$$

and the time dependent inhomogeneous terms are:  $L_q(\tau) = 2q[\tau + \sqrt{\Gamma^2 - 1} \cos \tau - \sin \tau]$ ,  $M_q(\tau) = -2q^2(\Gamma^2 - 1) \cos \tau + 2q^2\sqrt{\Gamma^2 - 1} \sin \tau + \frac{q^2(\Gamma^2 - 2)}{2} \cos(2\tau) - q^2\sqrt{\Gamma^2 - 1} \sin(2\tau) - 2q^2\sqrt{\Gamma^2 - 1}\tau(1 - \cos \tau) - 2q^2\tau \sin \tau + q^2\tau^2$  and  $N_q(\tau) = -q\sqrt{\Gamma^2 - 1} \sin \tau - q \cos \tau + \hat{R}^{-1}q(q^2 + \hat{\pi}^2 m^2)(\tau + \sqrt{\Gamma^2 - 1} \cos \tau - \sin \tau)$ .

Note that the equation (7) is valid only between  $\tau = 2n\pi$  and  $\tau = \tau_0 + 2n\pi$  with  $\tau_0 \equiv t_1 - t_0$ , during which grains are launched from the plate by vibrations and then undergo free fall. Except for this region, it is easy to show  $S(t) = 0$  and  $C(q) = E(q) = L_q(\tau) = M_q(\tau) = N_q(\tau) = 0$  with  $D(q) \rightarrow D_0(q) = T_e(q^2 + \hat{\pi}^2 m^2)$ .

The rest of the paper is devoted to discuss the condition for the linear stability of (7) and numerically test the validity of such approximations. We may be able to obtain an explicit solution of Eq.(7) with the aid of assumption that the most unstable mode is only

a relevant mode. The condition for instability from this treatment, however, is complicated and time dependent. In addition, this condition is not adequate for our purpose, since we are interested in the behavior in time longer than one vibrating oscillation. Therefore, we may replace  $L_q(\tau)$  (and for  $M_q(\tau)$  and  $N_q(\tau)$  as well) by its average value over a *flying time*  $< L_q(\tau) >$ , namely  $L_q(\tau) \simeq < L_q > = \frac{1}{\tau_0} \int_0^{\tau_0} d\tau L_q(\tau)$  and so on for  $M_q(\tau)$  and  $N_q(\tau)$ . Eq.(7) is then reduced to a second order ordinary differential equation with constant coefficients. Assuming  $\rho_q \sim e^{\sigma t}$ , it becomes easy to obtain the eigenvalues  $\sigma$  for the flying motion as

$$\sigma_{\pm} = -\frac{B + i\tilde{C}}{2} \pm \frac{1}{2} \sqrt{(B + i\tilde{C})^2 - 4(\tilde{D} + i\tilde{E})} \quad (10)$$

where  $\tilde{C} = C(q) - < L_q >$ ,  $\tilde{D} = D(q) - < M_q >$  and  $\tilde{E} = E(q) - < N_q >$ . The relevant branch is only  $\sigma_+$ . On the other hand, the eigenvalues are reduced to  $\sigma_{\pm} = -B/2 \pm \sqrt{B^2 - 4D_0}/2$  for stationary states.

The averaged instability condition over one oscillation cycle is then the average of  $Re[\sigma] > 0$ . For this purpose, we introduce a function:

$$\sigma_{eff}(q) = \tau_0 \left\{ \left( \tilde{E} - \frac{B\tilde{C}}{2} \right)^2 - B^2 \left( \tilde{D} + \frac{\tilde{C}^2}{4} \right) \right\} + (2\pi - \tau_0)(-B^2 D_0). \quad (11)$$

where the first term is the instability condition for (10) multiplied by the time period,  $\tau_0$ , in which particles can move freely [14], while the second term is that with  $S(t) = 0$ . If the function  $\sigma_{eff}(q) > 0$  for any  $q$ , it signals the instability of the uniform bouncing solution. For finite system,  $\sigma_{eff}(0) = -2\pi^7 T_e / (\hat{R}^2 L^6) < 0$ . Thus, the convection will disappear for infinite systems, which agrees with MD simulations [1,3,15]. Equivalently, convection also disappears in the limit of large  $R$ , i.e. either the particles are too smooth or the kinetic energy is too small to provide the necessary driving force among grains. The set of parameters that corresponds to physical situations might be:  $\hat{R} \sim 2$ ,  $T_e \sim 3$  and  $L = 10$ , because (i) the linear size of the box  $L_r = Lg/\omega^2 \simeq 0.6[cm]$  for  $\omega \simeq 20[Hz]$ , (ii)  $T \sim \tau_0^{-1} \int_0^{\tau_0} V_{rel}^2(\tau) d\tau \sim 3$ , (iii) the kinetic viscosity for granular fluid is evaluated by  $\nu_s \simeq 5 \times 10^{-3}[m^2/s]$  [2] and the definition of  $R = UL_r/\nu_s \sim 2$  with the aid of the characteristic velocity  $U \sim \sqrt{V_{rel}^2}g/\omega \sim 10cm/sec$  in the physical unit. But for pure numerical reasons, we choose  $\hat{R} = T_e = 10$  and  $L = 10$ .

For this set of parameters, we first solve  $\Delta(t) = 0$  numerically to determine  $t_1$ , and then compute  $\sigma_{eff}(q)$  as a function of  $q$ . As demonstrated in Fig.1,  $\sigma_{eff}(q)$  is convex and thus has a maximum,  $\sigma_m$ , at a particular value of  $q$ . For  $\Gamma \simeq 1$ ,  $\sigma_m < 0$  and thus  $\sigma_{eff}(q) < 0$  for all  $q$  and the bouncing solution is stable. But as we increase  $\Gamma$  further to the critical value,  $\Gamma_c$ ,  $\sigma_m$  moves upward crossing zero and becomes positive, in which case  $\sigma_{eff}(q) > 0$  for a band of  $q$ . In this case, the bouncing solution becomes unstable and we expect the convective rolls to appear. The onset of convection is then determined by setting,  $\sigma_m(\Gamma_c) = 0$ . For  $L = 10, R = T_e = 10$ , we find  $\Gamma_c \simeq 1.12$  and the selected wave number is about  $q_c = 0.22$ . The most unstable wave number  $q_m$  gradually shifts with the increase of  $\Gamma$ . We now check the validity of our picture by numerical simulations.

**Numerical Results:** We have solved (2)-(4) numerically in two dimension with no slip boundary conditions at the side walls as well as at the top and the bottom plates. Note that the top plate suppress complicated surface motion of vibrating beds and allow us to use the simplified picture. Since the granular fluid is confined in a box, we do not introduce  $\lambda$  explicitly in the simulations. As a result,  $S(t) \approx 0$  after a grain lands on a plate in the average bouncing state. The absence of  $\lambda$  and the presence of the top wall is expected to cause the appearance of the bouncing solution for  $\Gamma \leq 1$  in contrast with the real situation. But since the linearized eq.(7) with  $S(t) = 0$  is identical to that with non-zero  $\lambda$ , omitting  $\lambda$  would not change the essence of the dynamics. In the same spirit, we have ignored  $\chi$  and  $b$  in our simulations. Our simulation results are presented in Fig.2 for two different values of  $\Gamma$ ,  $\Gamma < \Gamma_c$  and  $\Gamma > \Gamma_c$ . In the former case, the bouncing solution is expected to appear inside the bed and the density and the velocity at a given point oscillates with the same frequency of the vibration.(Fig.2a) Upon increasing  $\Gamma$  further to  $\Gamma = 1.2$ , which is beyond the predicted  $\Gamma_c = 1.12$  determined by (11), we find that the bouncing solution has disappeared and the permanent convective rolls have developed inside the bulk (Fig.2b). The wavelength of the most unstable mode by the linear stability analysis is about  $q_m \approx 0.4$ , which is not far from the actual wavelength of the convective rolls:  $q = 2\pi/\lambda = 2\pi/L \approx 0.6$ .

In passing, we briefly mentioned the difference between the granular beds and the water beds. The latter is shown to exhibit the Faraday instability on the air-water interface. [16] The crucial difference between these two systems lie in the pressure term: for the water bed, since the water is an incompressible fluid, the hydrodynamic pressure term  $\rho g z$  precisely cancels the gravity term in the fluid equation, thus suppressing the motion inside the bulk, while the absence of the hydrodynamic pressure term produces the convective instability in the bulk for the granular beds. We will present the details of our analysis including the weakly nonlinear analysis elsewhere, which will highlight the differences between the two.

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## Figure Captions

Fig.1. The effective growth rate  $\sigma_{eff}(q)$  as a function of the wave number  $q$  for  $\Gamma = 1.05$ (diamond), for which  $\sigma_{eff}(q) < 0$  for all values of  $q$ , while for  $\Gamma = 1.2 > \Gamma_c = 1.12$ ,  $\sigma_{eff}(q)$  becomes positive for a band of  $q$ (square).  $\Gamma_c$  is determined by the condition that the maximum of  $\sigma_{eff}(q)$  becomes zero at  $\Gamma_c$ .(cross) The parameters used are:  $T_e = R = 10$  and  $L = 10$ .

Fig.2 (a) A bouncing solution. The speed  $v_z$  at a given point is plotted as a function time for  $\Gamma = 0.9$ . (b) For  $\Gamma = 1.2 > \Gamma_c = 1.12$ , the bouncing solution becomes unstable and the permanent convective rolls appear inside the box. The arrows are the velocity vectors pointing upward. The parameters used in simulations for (a) and (b) are the same as those in Fig.1.





